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GENERAL UNREPLICATED LINEAR FUNCTIONAL RELATIONSHIP MODEL FOR CIRCULAR VARIABLES WITH WIND DIRECTION APPLICATION

Mohd Faisal Saari[a*], Abdul Ghapor Hussin[b], Mohd Syazwan Mohamad Anuar[a], Yong Zulina Zubairi[c]

[a] Centre for Defence Foundation Studies, National Defence University of Malaysia, Sg. Besi Camp, 57000 Kuala Lumpur, Malaysia
[b] Department of Defence Science, Faculty of Defence Science and Technology, National Defence University of Malaysia, Sg. Besi Camp, 57000 Kuala Lumpur, Malaysia
[c] Centre for Foundation Studies in Science, Universiti Malaya, 50603 Kuala Lumpur, Malaysia

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ABSTRACT

The functional relationship model is typically used to describe the nature of data that contains unobservable errors. The general unreplicated linear functional relationship model is discussed in this paper, which can be used to examine circular data with measurement errors. All factors involved in the unreplicated linear functional relationship model, such as the rotation parameter, slope parameter, and concentration parameter for both observed variables, where the ratio is known and can be equal or unequal, will be evaluated. This model's parameter estimation is somewhat difficult; however, it is possible to achieve numerical results using a simple iteration technique. To validate, analyse, and investigate the model's performance, a simulation study was constructed utilizing Monte Carlo simulation. The results reveal that, in general, estimation bias is minimal and acceptable. The concept is demonstrated using an application to the analysis of a real-world wind direction data collection.

1.0 INTRODUCTION

Data can be classified as linear data or circular data. Linear data is commonly used daily and this type of data can be analysed by using the standard procedure of statistical technique. A circular data is a set of collection points on a circle of unit degree or radius [1]. The concept of circular data analysis can be traced back to the mid-18th century when the development of statistical models for circular response variables was discussed. Various scientific fields, such as life science [2], biology [3], phenology and environmental study [4], have been benefited from the study of circular data. Due to the nature of the angle, formal analysis of circular data cannot be done with the usual statistical technique.

The functional relationship model is part of the error-in-variables model (EIVM), which has deterministic or fixed underlying variables. The structural relationship model and the ultrastructural relationship model are two further models in EIVM. The structural relationship model is used when the variables are random [5]. Meanwhile, because the ultrastructural relationship model is a synthesis of the linear and structural relationship models, it contains both random and fixed variables [6].

The following are some of the key distinctions between conventional regression and EIVM. To begin, standard regression assumes that x value is mathematically observed without error and only y is seen with error for each pair of observations (x, y), but EIVM implies that both variables are observed with error. Second, unlike traditional linear regression, there is no separation between explanatory and response variables in EIVM. Finally, if the goal is to predict one variable from the other rather than

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

examine the underlying relationship between the two variables, regular linear regression is more appropriate.

Commonly when dealing with linear variables, a linear functional relationship model can be used to represent the underlying relationship between the variables. The same goes for circular or directional variables, which is the circular functional relationship model that will be used. Compared to linear functional, circular functional is more complex since there involve trigonometric expression in the model. In order to simplify the relationship, a linear functional relationship model still can be used to represent the circular variables. By modifying the regression model for circular variables, the functional relationship model for circular variables was first proposed [7]. Then it had been improved according to certain conditions and limitations such as slope parameter is considered fixed to one [8], the ratio of error concentration parameter is fixed to one [9] and simple functional model with unequal error concentration [10]. Many fields of expertise will be benefited from this general model such as business and economy [11] property sector [12] and agricultural water management [13].

This paper aims to propose a general unreplicated linear functional relationship model for circular variables by trying to eliminate the previous condition and limitation. The proposed model will be applied to the real wind directional data collected from Holderness Coastline.

2.0 THE GENERAL UNREPLICATED LINEAR FUNCTIONAL RELATIONSHIP MODEL

Suppose x_i and y_i are observed values of the circular variables X and Y respectively, thus $0 \le x_i, y_i \le 2\pi$ for $i = 1,2,\ldots,n$. For any fixed X_i, assume that the observations x_i and y_i have been measured with errors δ_i and ε_i respectively. The data are concentrated on interval $[0,2\pi)$ and linear combination of angles should be in $[0,2\pi)$ also, thus modulo 2π needs to be added into the linear relationship between X and Y circular variables. With the assumption of α as rotation parameter and β as slope parameter, the general unreplicated linear functional relationship (LFRM) model can be written in the form of

$$x_i = X_i + \delta_i \text{ and } y_i = Y_i + \varepsilon_i \tag{1}$$

where

$$Y_i = \alpha + \beta X_i \left(\bmod 2\pi\right), \text{ for } i = 1,2,3,\ldots,n$$

δ_i and ε_i also assumed to be independently distributed by von Mises distributions, that is $\delta_i \sim VM(0,\kappa_x)$ and $\varepsilon_i \sim VM(0,\kappa_y)$ where κ_x is error concentration parameter for X variable and κ_y is error concentration parameter for Y variable.

Von Mises distribution is one of the best to describe the distribution on the circle [14]. For any circular random variables θ, with mean direction μ and concentration parameter κ, von Mises distribution probability density function is given by,

$$g(\mu,\kappa;\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left\{\kappa\cos(\theta-\mu)\right\}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, defined as $I_0(\kappa) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{\kappa\cos\theta\}\, d\theta$. The log-likelihood function of the von Mises distribution can be presented by:

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

$$\log L\left(\alpha,\beta,\kappa_x,\kappa_y,X_i;x_i,y_i\right)=$$
$$-2n\log\left(2\pi\right)-n\log I_0\left(\kappa_x\right)-n\log I_0\left(\kappa_y\right)+\kappa_x\sum_{i=1}^{n}\cos\left(x_i-X_i\right)+\kappa_y\sum_{i=1}^{n}\cos\left(y_i-\alpha-\beta X_i\right) \tag{2}$$

3.0 THE MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS

In Equation (2), there are $(n+4)$ parameters that need to be estimated. As for simplification of the estimation process, the ratio of concentration parameter λ was assumed as known and defined as $\lambda=\kappa_y/\kappa_x$. Hence, Equation (2) can be simplified into $(n+3)$ parameters to be estimated equation.

$$\log L\left(\alpha,\beta,\kappa_x,X_i;x_i,y_i,\lambda\right)=$$
$$-2n\log\left(2\pi\right)-n\log I_0\left(\kappa_x\right)-n\log I_0\left(\lambda\kappa_x\right)+\kappa_x\sum_{i=1}^{n}\cos\left(x_i-X_i\right)+\lambda\kappa_x\sum_{i=1}^{n}\cos\left(y_i-\alpha-\beta X_i\right) \tag{3}$$

Differentiating Equation (3) with respect to parameters α,β,κ_x and X_i, all estimated parameters can be obtained.

3.1 Parameter Estimation For Rotation Parameter, $\hat{\alpha}$

The first partial derivatives of Equation (3) with respect to α is

$$\frac{\partial\log L}{\partial\alpha}=\sum_{i=1}^{n}\sin\left(y_i-\alpha-\beta X_i\right)$$

Setting this equal to zero and simplifying

$$\sum_{i=1}^{n}\sin\left(y_i-\hat{\beta}\hat{X}_i\right)\cos\hat{\alpha}-\sum_{i=1}^{n}\cos\left(y_i-\hat{\beta}\hat{X}_i\right)\sin\hat{\alpha}=0$$

Solving for $\hat{\alpha}$

$$\tan\hat{\alpha}=\frac{\sum_{i=1}^{n}\sin\left(y_i-\hat{\beta}\hat{X}_i\right)}{\sum_{i=1}^{n}\cos\left(y_i-\hat{\beta}\hat{X}_i\right)}$$

$$\hat{\alpha}=\tan^{-1}\left[\frac{\sum_{i=1}^{n}\sin\left(y_i-\hat{\beta}\hat{X}_i\right)}{\sum_{i=1}^{n}\cos\left(y_i-\hat{\beta}\hat{X}_i\right)}\right]$$

Let $\hat{\alpha}=\tan^{-1}\left(\frac{S}{C}\right)$, where $S=\sum_{i=1}^{n}\sin\left(y_i-\hat{\beta}\hat{X}_i\right)$ and $C=\sum_{i=1}^{n}\cos\left(y_i-\hat{\beta}\hat{X}_i\right)$. Then

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right) & S > 0, C > 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + \pi & C < 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + 2\pi & S < 0, C > 0 \end{cases} \tag{4}$$

3.2 Parameter Estimation For Slope Parameter, $\hat{\beta}$

The first partial derivatives of Equation (3) with respect to β is

$$\frac{\partial \log L}{\partial \beta} = \lambda \kappa_x \sum_{i=1}^{n} X_i \sin\left(y_i - \alpha - \beta X_i\right)$$

Setting this equal to zero and simplifying

$$\sum_{i=1}^{n} \hat{X}_i \sin\left(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i\right) = 0$$

$\hat{\beta}$ can be obtained iteratively. Suppose $\hat{\beta}_0$ is an initial estimation of $\hat{\beta}$. Using the Newton-Raphson iteration method to estimate $\hat{\beta}$ gives

$$\hat{\beta}_1 = \hat{\beta}_0 + \frac{\sum_{i=1}^{n} \hat{X}_i \sin\left(y_i - \hat{\alpha} - \hat{\beta}_0 \hat{X}_i\right)}{\sum_{i=1}^{n} \hat{X}_i^{\,2} \cos\left(y_i - \hat{\alpha} - \hat{\beta}_0 \hat{X}_i\right)} \tag{5}$$

where $\hat{\beta}_1$ is an improvement of $\hat{\beta}_0$.

3.3 Parameter Estimation For Concentration Parameter, $\hat{\kappa}$

The first partial derivatives of Equation (3) with respect to κ_x is

$$\frac{\partial \log L}{\partial \kappa_x} = -n\frac{I_0'\left(\kappa_x\right)}{I_0\left(\kappa_x\right)} - n\frac{I_0'\left(\lambda\kappa_x\right)}{I_0\left(\lambda\kappa_x\right)} + \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \lambda\sum_{i=1}^{n}\cos\left(y_i - \alpha - \beta X_i\right)$$

$$\frac{\partial \log L}{\partial \kappa_x} = -nA\left(\kappa_x\right) - n\lambda A\left(\lambda\kappa_x\right) + \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \lambda\sum_{i=1}^{n}\cos\left(y_i - \alpha - \beta X_i\right) \tag{6}$$

where $A\left(\kappa_x\right)$ and $A\left(\lambda\kappa_x\right)$ are the ratio of the modified Bessel function of the first kind and order one and the first kind and order zero, defined as

$$A\left(\kappa_x\right) = \frac{I_0'\left(\kappa_x\right)}{I_0\left(\kappa_x\right)} = \frac{I_1\left(\kappa_x\right)}{I_0\left(\kappa_x\right)} = 1 - \frac{1}{2\kappa_x} - \frac{1}{8\kappa_x^{\,2}} - \frac{1}{8\kappa_x^{\,3}}$$

$$A\left(\lambda\kappa_x\right) = \frac{I_0'\left(\lambda\kappa_x\right)}{I_0\left(\lambda\kappa_x\right)} = \frac{I_1\left(\lambda\kappa_x\right)}{I_0\left(\lambda\kappa_x\right)} = 1 - \frac{1}{2\left(\lambda\kappa_x\right)} - \frac{1}{8\left(\lambda\kappa_x\right)^2} - \frac{1}{8\left(\lambda\kappa_x\right)^3}$$

Setting Equation (6) to zero and simplifying

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

$$-nA\left(\hat{\kappa}_x\right)-n\lambda A\left(\lambda\hat{\kappa}_x\right)+\sum_{i=1}^{n}\cos\left(x_i-\hat{X}_i\right)+\lambda\sum_{i=1}^{n}\cos\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_i\right)=0$$

$$A\left(\hat{\kappa}_x\right)+\lambda A\left(\lambda\hat{\kappa}_x\right)=\frac{1}{n}\left[\sum_{i=1}^{n}\cos\left(x_i-\hat{X}_i\right)+\lambda\sum_{i=1}^{n}\cos\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_i\right)\right]$$

Let $w=\frac{1}{n}\left[\sum_{i=1}^{n}\cos\left(x_i-\hat{X}_i\right)+\lambda\sum_{i=1}^{n}\cos\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_i\right)\right]$

$$A\left(\hat{\kappa}_x\right)+\lambda A\left(\lambda\hat{\kappa}_x\right)=\left(1-\frac{1}{2\hat{\kappa}_x}-\frac{1}{8\hat{\kappa}_x^{\,2}}-\frac{1}{8\hat{\kappa}_x^{\,3}}\right)+\lambda\left(1-\frac{1}{2\left(\lambda\hat{\kappa}_x\right)}-\frac{1}{8\left(\lambda\hat{\kappa}_x\right)^2}-\frac{1}{8\left(\lambda\hat{\kappa}_x\right)^3}\right)=w$$

Simplifying the above equation to become a cubic equation of

$$8\left(1+\lambda-w\right)\hat{\kappa}_x^{\,3}-8\hat{\kappa}_x^{\,2}-\left(1+\frac{1}{\lambda}\right)\hat{\kappa}_x-\left(1+\frac{1}{\lambda^2}\right)=0 \tag{7}$$

However, the estimation of κ_x in Equation (7) cannot be solved directly. To solve this cubic expression, the build-in numerical approximation function called *polyroot* function in R software can be used. This will give a real root and two complex roots for which the real root will be chosen as an approximation κ_x, $\tilde{\kappa}_x$. The correction factor raised by Caires and Wyatt [15] for estimation of concentration parameter was applied for this case which will give the estimation of κ_x as $\hat{\kappa}_x=\tilde{\kappa}_x/2$.

As for concentration parameter for Y, since λ is known κ_y can be estimated by using the definition of ratio concentration parameter, which gives $\hat{\kappa}_y=\lambda\hat{\kappa}_x$.

3.4 Parameter Estimation For Incidental Parameters, $\hat{X}_i$

The first partial derivatives of Equation (3) with respect to X_i is

$$\frac{\partial\log L}{\partial X_i}=\kappa_x\sum_{i=1}^{n}\sin\left(x_i-X_i\right)+\lambda\kappa_x\beta\sum_{i=1}^{n}\sin\left(y_i-\alpha-\beta X_i\right)$$

Setting this equal to zero and simplifying

$$\sin\left(x_i-\hat{X}_i\right)+\lambda\hat{\beta}\sin\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_i\right)=0$$

As $\hat{\beta}$, $\hat{X}_i$ can be obtained iteratively. Suppose $\hat{X}_{i0}$ is an initial estimation of $\hat{X}_i$. Using the Newton-Raphson iteration method to estimate $\hat{X}_i$ gives

$$\hat{X}_{i1}=\hat{X}_{i0}+\frac{\sin\left(x_i-\hat{X}_{i0}\right)+\lambda\hat{\beta}\sin\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_{i0}\right)}{\cos\left(x_i-\hat{X}_{i0}\right)+\lambda\hat{\beta}^2\cos\left(y_i-\hat{\alpha}-\hat{\beta}\hat{X}_{i0}\right)} \tag{8}$$

where $\hat{X}_{i1}$ is an improvement of $\hat{X}_{i0}$.

3.5 The Variance Of The Parameters

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

By using various approximations [16] and Fisher information matrix [17], the estimated variance of parameters can be obtained. For any values for the ratio of concentration parameter λ, it can be shown that

$$V\hat{a}r\left(\hat{\alpha}\right)=\frac{\left[\hat{\kappa}_x A\left(\hat{\kappa}_x\right)+\lambda\hat{\kappa}_x\hat{\beta}^2 A\left(\lambda\hat{\kappa}_x\right)\right]\sum_{i=1}^{n}\hat{X}_i^2}{\hat{\kappa}_x A\left(\hat{\kappa}_x\right)\lambda\hat{\kappa}_x A\left(\lambda\hat{\kappa}_x\right)\left[n\sum_{i=1}^{n}\hat{X}_i^2-\left(\sum_{i=1}^{n}\hat{X}_i\right)^2\right]}$$

$$V\hat{a}r\left(\hat{\beta}\right)=\frac{n\left[\hat{\kappa}_x A\left(\hat{\kappa}_x\right)+\lambda\hat{\kappa}_x\hat{\beta}^2 A\left(\lambda\hat{\kappa}_x\right)\right]}{\hat{\kappa}_x A\left(\hat{\kappa}_x\right)\lambda\hat{\kappa}_x A\left(\lambda\hat{\kappa}_x\right)\left[n\sum_{i=1}^{n}\hat{X}_i^2-\left(\sum_{i=1}^{n}\hat{X}_i\right)^2\right]}$$

$$V\hat{a}r\left(\hat{\kappa}_x\right)=\frac{\hat{\kappa}_x}{n}\left\{\frac{1}{\left[\hat{\kappa}_x-\hat{\kappa}_x A^2\left(\hat{\kappa}_x\right)-A\left(\hat{\kappa}_x\right)\right]+\lambda\left[\lambda\hat{\kappa}_x-\lambda\hat{\kappa}_x A^2\left(\lambda\hat{\kappa}_x\right)-A\left(\lambda\hat{\kappa}_x\right)\right]}\right\}$$

4.0 SIMULATION STUDY

In order to evaluate the accuracy of the parameters in this proposed model, a Monte Carlo simulation study was performed using the software R. The number of simulations s is set to be 5000 for each set of simulations. Without loss of generality, let the true value of $\alpha=\pi/4$, $\beta=1$ and $\lambda=0.8, 1.0, 1.2$ (to represent the equal and unequal error of concentration parameter), meanwhile the corresponding value of κ_x and κ_y are shown in Table 1. In order to simulate the realistic range of error concentration parameter, $\kappa_x=5, 10, 15$ was chosen since circular variables are less dispersed compared to linear variables. The set of X variable has been generated from the von Mises distribution where $X\sim VM(\pi/4, 5)$ and the sample size $n=50, 100, 200, 500$ are considered for the simulation.

Table 1. Values of κ_x and κ_y for each λ

λ	κ_x	κ_y
0.8	5	4
	10	8
	15	12
1.0	5	5
	10	10
	15	15
1.2	5	6
	10	12
	15	18

4.1 Biasness Of α

In this model, α is considered the circular parameter. Therefore $\hat{\alpha}$ can be verified by using three measures.

i. Circular mean

$$\bar{\hat{\alpha}}=\begin{cases}\tan^{-1}\left(\frac{S}{C}\right) & S>0, C>0\\ \tan^{-1}\left(\frac{S}{C}\right)+\pi & C<0\\ \tan^{-1}\left(\frac{S}{C}\right)+2\pi & S<0, C>0\end{cases}$$

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

where $S=\sum_{j=1}^{s}\sin\left(\hat{\alpha}_j\right)$ and $C=\sum_{j=1}^{s}\cos\left(\hat{\alpha}_j\right)$

ii. Circular distance, $d=\pi-\left|\pi-\left|\bar{\hat{\alpha}}-\alpha\right|\right|$

iii. Mean resultant length, $\bar{R}=\frac{1}{s}\sqrt{\left(\sum_{j=1}^{s}\cos\left(\hat{\alpha}_j\right)\right)^2+\left(\sum_{j=1}^{s}\sin\left(\hat{\alpha}_j\right)\right)^2}$

4.2 Biasness Of β,κ_x And κ_y

It is noted that β,κ_x *and* κ_y is considered the continuous parameter for this model. Therefore, these estimated parameters can be tested by using three measures. Let ω be a generic term for β,κ_x and κ_y. Then

i. Mean, $\bar{\hat{\omega}}=\frac{1}{s}\sum_{j=1}^{s}\hat{\omega}_j$

ii. Estimated bias, $EB=\bar{\hat{\omega}}-\omega$

iii. Estimated root mean square errors, $ERMSE=\sqrt{\frac{1}{s}\sum_{j=1}^{s}\left(\hat{\omega}_j-\omega\right)^2}$

5.0 SIMULATION RESULT AND DISCUSSION

Focusing on mean resulting length and ERMSE, the average value for each group depending on sample size and concentration parameters can be plotted as shown in Figure 1 and Figure 2. From Table 2 it appears that $\hat{\alpha}$ is a good α estimator since the value of the circular mean approaches the true value (0.7854 rad or $\pi/4$) when n is increased for any value of κ. In general, when sample size and concentration parameters increase, then the circular distance, d which represents the biasness of $\hat{\alpha}$ decreases. From Figure 1, the mean resultant length, $\bar{R}$ also suggests good accuracy as the value is close to one.

Table 2. Simulation result for $\hat{\alpha}$

λ	κ	n	Performance indicator		
			Circular mean	Circular distance	Mean resultant length
$\lambda=0.8$	$\kappa_x=5$	50	0.7713	0.0141	0.9798
	$\kappa_y=4$	100	0.7774	0.0080	0.9902
		200	0.7796	0.0058	0.9953
		500	0.7825	0.0029	0.9981
	$\kappa_x=10$	50	0.7799	0.0055	0.9909
	$\kappa_y=8$	100	0.7818	0.0036	0.9957
		200	0.7825	0.0029	0.9978
		500	0.7839	0.0015	0.9992
	$\kappa_x=15$	50	0.7809	0.0045	0.9941
	$\kappa_y=12$	100	0.7824	0.0030	0.9971
		200	0.7832	0.0022	0.9986
		500	0.7840	0.0014	0.9995
$\lambda=1.0$	$\kappa_x=5$	50	0.7684	0.0170	0.9828
	$\kappa_y=5$	100	0.7728	0.0126	0.9918
		200	0.7781	0.0073	0.9960
		500	0.7827	0.0027	0.9985
	$\kappa_x=10$	50	0.7753	0.0100	0.9922
	$\kappa_y=10$	100	0.7800	0.0054	0.9962
		200	0.7823	0.0031	0.9982
		500	0.7840	0.0014	0.9993
	$\kappa_x=15$	50	0.7787	0.0067	0.9949

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

λ	κ	n	Performance indicator		
			Circular mean	Circular distance	Mean resultant length
		100	0.7806	0.0048	0.9976
		200	0.7822	0.0032	0.9988
		500	0.7844	0.0010	0.9995
$\lambda = 1.2$	$\kappa_x = 5$, $\kappa_y = 6$	50	0.7667	0.0187	0.9852
		100	0.7741	0.0113	0.9929
		200	0.7797	0.0057	0.9965
		500	0.7819	0.0035	0.9986
	$\kappa_x = 10$, $\kappa_y = 12$	50	0.7736	0.0118	0.9928
		100	0.7788	0.0066	0.9967
		200	0.7816	0.0038	0.9984
		500	0.7842	0.0012	0.9994
	$\kappa_x = 15$, $\kappa_y = 18$	50	0.7807	0.0047	0.9954
		100	0.7816	0.0038	0.9978
		200	0.7827	0.0027	0.9989
		500	0.7845	0.0008	0.9996

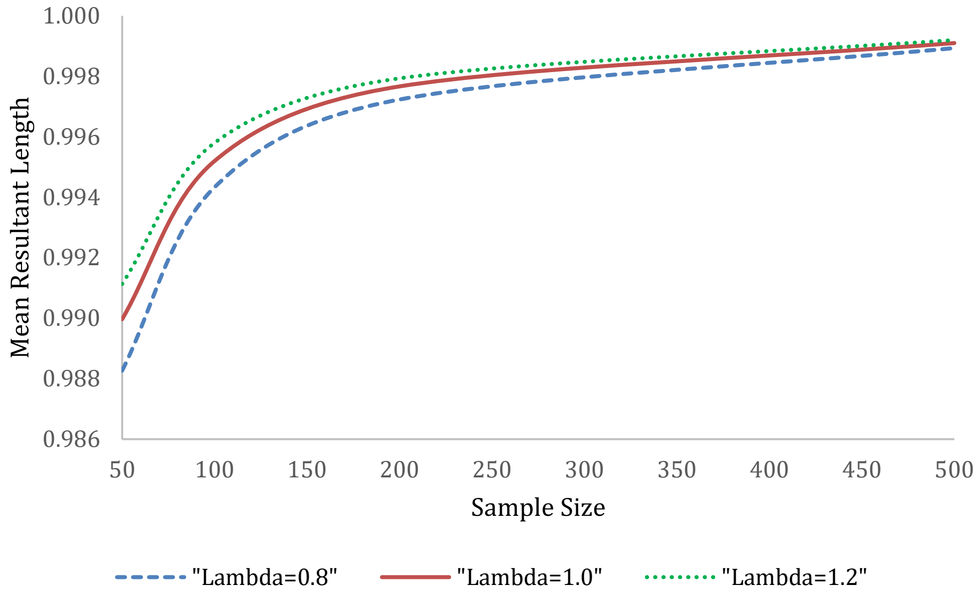


Figure 1. Mean resultant length for $\hat{\alpha}$

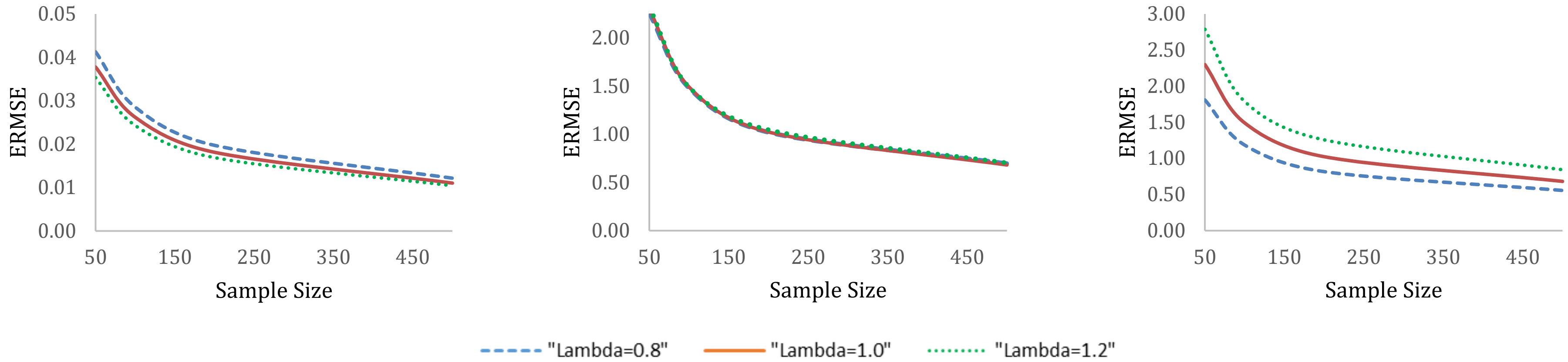
*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

Table 3. Simulation result for $\hat{\beta}, \hat{\kappa}_x$ and $\hat{\kappa}_y$

λ	κ	n	Performance indicator for $\hat{\beta}$			Performance indicator for $\hat{\kappa}_x$			Performance indicator for $\hat{\kappa}_y$		
			Mean	Estimated Bias	ERMSE	Mean	Estimated Bias	ERMSE	Mean	Estimated Bias	ERMSE
		50	1.0039	0.0039	0.0563	5.3633	0.3633	1.0593	4.2907	0.2907	0.8475
	$\kappa_x = 5$	100	1.0030	0.0030	0.0390	5.3073	0.3073	0.7109	4.2458	0.2458	0.5687
	$\kappa_y = 4$	200	1.0017	0.0017	0.0266	5.2218	0.2218	0.5613	4.1775	0.1775	0.4490
		500	1.0010	0.0010	0.0165	5.0398	0.0398	0.4681	4.0318	0.0318	0.3745
		50	1.0019	0.0019	0.0377	10.4710	0.4710	2.2440	8.3768	0.3768	1.7952
$\lambda = 0.8$	$\kappa_x = 10$	100	1.0013	0.0013	0.0255	10.2453	0.2453	1.4310	8.1963	0.1963	1.1448
	$\kappa_y = 8$	200	1.0011	0.0011	0.0182	10.1193	0.1193	1.0003	8.0955	0.0955	0.8002
		500	1.0003	0.0003	0.0110	10.0591	0.0591	0.6618	8.0473	0.0473	0.5295
		50	1.0013	0.0013	0.0298	15.9491	0.9491	3.4819	12.7592	0.7592	2.7855
	$\kappa_x = 15$	100	1.0012	0.0012	0.0211	15.3212	0.3212	2.3027	12.2570	0.2570	1.8422
	$\kappa_y = 12$	200	1.0005	0.0005	0.0144	15.2201	0.2201	1.5024	12.1761	0.1761	1.2019
		500	1.0004	0.0004	0.0090	15.0239	0.0239	0.9574	12.0191	0.0191	0.7659
		50	1.0052	0.0052	0.0511	5.3280	0.3280	1.0790	5.3280	0.3280	1.0790
	$\kappa_x = 5$	100	1.0039	0.0039	0.0354	5.2592	0.2592	0.7371	5.2592	0.2592	0.7371
	$\kappa_y = 5$	200	1.0018	0.0018	0.0242	5.1793	0.1793	0.5650	5.1793	0.1793	0.5650
		500	1.0010	0.0010	0.0150	5.0418	0.0418	0.4522	5.0418	0.0418	0.4522
		50	1.0030	0.0030	0.0346	10.5658	0.5658	2.2946	10.5658	0.5658	2.2946
$\lambda = 1.0$	$\kappa_x = 10$	100	1.0015	0.0015	0.0237	10.2216	0.2216	1.4442	10.2216	0.2216	1.4442
	$\kappa_y = 10$	200	1.0010	0.0010	0.0166	10.1125	0.1125	0.9859	10.1125	0.1125	0.9859
		500	1.0004	0.0004	0.0099	10.1005	0.1005	0.6491	10.1005	0.1005	0.6491
		50	1.0019	0.0019	0.0276	15.9830	0.9830	3.5226	15.9830	0.9830	3.5226
	$\kappa_x = 15$	100	1.0010	0.0010	0.0195	15.3569	0.3569	2.2941	15.3570	0.3570	2.2941
	$\kappa_y = 15$	200	1.0008	0.0008	0.0136	15.1233	0.1233	1.5210	15.1233	0.1233	1.5210
		500	1.0004	0.0004	0.0082	15.0535	0.0535	0.9445	15.0535	0.0535	0.9445
		50	1.0047	0.0047	0.0469	5.3779	0.3779	1.1227	6.4535	0.4535	1.3472
	$\kappa_x = 5$	100	1.0033	0.0033	0.0324	5.2972	0.2972	0.7895	6.3567	0.3567	0.9474
	$\kappa_y = 6$	200	1.0014	0.0014	0.0227	5.1943	0.1943	0.6208	6.2332	0.2332	0.7450
		500	1.0010	0.0010	0.0140	5.0121	0.0121	0.5070	6.0145	0.0145	0.6084
	$\kappa_x = 10$	50	1.0027	0.0027	0.0327	10.5519	0.5519	2.3242	12.6623	0.6623	2.7890
$\lambda = 1.2$	$\kappa_y = 12$	100	1.0017	0.0017	0.0224	10.1742	0.1742	1.4530	12.2090	0.2090	1.7436
		200	1.0012	0.0012	0.0154	10.1364	0.1364	0.9977	12.1637	0.1637	1.1973
		500	1.0004	0.0004	0.0096	10.0748	0.0748	0.6461	12.0898	0.0898	0.7753
		50	1.0018	0.0018	0.0264	15.9214	0.9214	3.5218	19.1056	1.1056	4.2262

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

λ	κ	n	Performance indicator for $\hat{\beta}$			Performance indicator for $\hat{\kappa}_x$			Performance indicator for $\hat{\kappa}_y$		
			Mean	Estimated Bias	ERMSE	Mean	Estimated Bias	ERMSE	Mean	Estimated Bias	ERMSE
	$\kappa_x = 15$	100	1.0013	0.0013	0.0181	15.3947	0.3947	2.2297	18.4736	0.4736	2.6757
	$\kappa_y = 18$	200	1.0009	0.0009	0.0126	15.1070	0.1070	1.5235	18.1285	0.1285	1.8282
		500	1.0003	0.0003	0.0077	15.0570	0.0570	0.9561	18.0684	0.0684	1.1473



(a) (b) (c)

Figure 2. Estimated root mean square errors (ERMSE) for (a) $\hat{\beta}$, (b) $\hat{\kappa}_x$ and (c) $\hat{\kappa}_y$

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

A similar conclusion can be obtained by referring to Table 3 and focusing on parameter $\hat{\beta}, \hat{\kappa}_x$ and $\hat{\kappa}_y$, where it suggests that a good prediction for parameter $\hat{\beta}, \hat{\kappa}_x$ and $\hat{\kappa}_y$ has also been conducted based on these simulation findings. The estimated bias (EB) and estimated root mean square errors (ERMSE) for $\hat{\beta}, \hat{\kappa}_x$ and $\hat{\kappa}_y$ decrease and approach zero as the sample size and concentration parameter values grow, while the mean values for each parameter approach their actual value. Figure 2 confirms these findings by demonstrating a resemblance between three graphs that demonstrate that the values of ERMSE for all continuous parameters for this model will shrink and fade away as the sample size rises.

6.0 APPLICATION TO WIND DIRECTION

Wind direction data taken from Holderness Coastline on the Humberside Coast of the North Sea in the United Kingdom may be shown using the general unreplicated functional relationship model. With a sample size of 129, wind direction data was obtained by an HF radar system created by UK Rutherford and Appleton Laboratories, which will be referred to as the x variable. The data for variables y were collected using an anchored wave buoy. Table 4 present the example of a different combination of parameters depending on the value ratio of the concentration parameter, λ. As we know, this proposed model is subject to the known value of λ and can be applied to any possible value of parameters. For this case, compared to HF radar system, x and anchored buoy, y professionals involved should know the ratio of effectiveness between these two variables which bring to the known value of λ (depending on the situation and professional demand).

Table 4. Mean and variance for parameter estimates on wind direction data

λ	$\hat{\alpha}$ (Variance)	$\hat{\beta}$ (Variance)	$\hat{\kappa}_x$ (Variance)	$\hat{\kappa}_y$ (Variance)
0.6	0.1197 (0.0055)	0.9859 (0.0003)	15.6339 (1.9064)	9.3803 (0.6863)
0.8	0.1198 (0.0057)	0.9872 (0.0003)	12.7311 (1.2623)	10.1849 (0.8079)
1.0	0.0954 (0.0059)	0.9926 (0.0003)	11.0694 (0.9534)	11.0694 (0.9534)
1.2	0.0626 (0.0063)	0.9980 (0.0004)	9.4841 (0.6978)	11.3810 (1.0049)
1.4	0.0752 (0.0067)	0.9953 (0.0004)	8.6305 (0.5771)	12.0827 (1.1312)
2.0	0.0587 (0.0109)	0.9941 (0.0006)	4.8299 (0.1762)	9.6598 (0.70475)

For example, let's say $\lambda = 1.2$ is chosen. Then the relationship between wind direction variables can be written as $Y = 0.0626 + 0.998X \left(\operatorname{mod} 2\pi\right)$ where $\delta_i : VM\left(0, 9.4841\right)$ and $\varepsilon_i : VM\left(0, 11.381\right)$. Since $V\hat{a}r\left(\hat{\alpha}\right) = 0.0063$, $V\hat{a}r\left(\hat{\beta}\right) = 0.0004$, $V\hat{a}r\left(\hat{\kappa}_x\right) = 0.6978$ and $V\hat{a}r\left(\hat{\kappa}_y\right) = 1.0049$ are a small value means that it indicates good estimation for the expected parameters.

7.0 CONCLUSION

This paper proposes the general unreplicated linear functional relationship model involving circular variables. The proposed model also considers all possible parameters, and all parameters can vary for all values depending on the condition for each parameter. Parameter estimation has been obtained by the maximum likelihood method. Based on the Monte Carlo simulation study, it suggested that the parameter estimation gives a good and consistent estimate since the mean of expected estimation is close to the true value and the bias becomes smaller as the sample size and concentration parameter is increasing. By using various approximations and the Fisher Information matrix, the Variance-Covariance matrix of the estimated parameters can be obtained. The model was applied to real data collected from the Holderness Coastline by examining the relationship between wind direction and two different measurements (HF radar system and anchored wave buoy). It is discovered that the suggested model describes the

*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com

underlying link between the measurement of two circular variables by assuming a known ratio of error concentration parameter. Based on the simulation results, it is reasonable to conclude that the proposed model is reliable for modelling circular data in general, with next to no bias. Unlike previous studies of unreplicated functional models by Hassan et al. (2010) and Mokhtar et al. (2015), this proposed unreplicated linear functional relationship model can estimate the parameters without assuming and fulfilling any condition or limitation, and it considers all parameters involved. This characteristic is the model's strength if the ratio of error concentration parameter is known.

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*Corresponding Author | Saari, M. F. | mohdfaisal@gmail.com